Int. J. Solids Structures Vol. 23, No. 12, pp. 1625–1638, 1987
Printed in Great Britain.

0020–7683/87 \$3.00+ .00

NON-LINEAR OSCILLATION OF CIRCULAR CYLINDRICAL SHELLS

ALI H. NAYFEH and RAOUF A. RAOUF
Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U.S.A.

(*Received* 18 *December* 1986; *in revised form* 31 *March* 1987)

Abstract—The method of multiple scales is used to analyze the non-linear forced response of circular cylindrical shells in the presence of a two-to-one internal (autoparametric) resonance to a harmonic excitation having the frequency Ω. If ω_f and a_f denote the frequency and amplitude of a flexural mode and ω_b and a_b denote the frequency and amplitude of the breathing mode, the steady-state response exhibits a saturation phenomenon when $\omega_b \approx 2\omega_f$ if the excitation frequency Ω is near ω_b. As the amplitude f of the excitation increases from zero, a_b increases linearly whereas a_f remains zero until a threshold is reached. This threshold is a function of the damping coefficients and $\omega_b - 2\omega_f$. Beyond this threshold a_b remains constant (i.e. the breathing mode saturates) and the extra energy spills over into the flexural mode. In other words, although the breathing mode is directly excited by the load, it absorbs a small amount of the input energy (responds with a small amplitude) and passes the rest of the input energy into the flexural mode (responds with a large amplitude). For small damping coefficients and depending on the detunings of the internal resonance and the excitation, the response exhibits a Hopf bifurcation and consequently there are no steady-state periodic responses. Instead, the responses are amplitude- and phase-modulated motions. When $\Omega \approx \omega_f$, there is no saturation phenomenon and at close to perfect resonance, the response exhibits a Hopf bifurcation, leading again to amplitude- and phase-modulated or chaotic motions.

1. INTRODUCTION

Recently, the problem of the non-linear vibration of shells has received considerable attention. The sources of the nonlinearities in the governing equations may be geometric, or inertial, or material, or any combination. The geometric nonlinearity stems from non-linear strain–displacement relations (e.g. mid-plane stretching, large curvatures and large rotations), the inertial nonlinearity may be caused by the presence of concentrated or distributed masses, and the material nonlinearity occurs when the stresses are non-linear functions of the strains. The nonlinearities appear in the governing partial-differential equations and they may appear in the boundary conditions. However, most of the existing studies of other than composite shells deal with geometric nonlinearities. Except for the studies of Dowell and Ventres[1] and Ginsberg[2], all existing studies deal with linear boundary conditions.

A number of non-linear governing equations have been proposed for the dynamic response of shells. They include the theories of Donnell[3], Novozhilov[4], Sanders[5] and Reissner[6]. The main differences among these theories are the approximations used in relating the strains and curvatures to the displacements. Donnell's theory is the most widely used of all these theories.

Since the problem is governed by partial-differential equations, the response, in general, consists of many modes. In fact, using the Galerkin procedure one obtains an infinite set of non-linear coupled equations describing the time variation of the amplitudes of the infinitely many modes. All existing studies truncate the infinite set of equations to a finite number and many of them keep only one mode.

Although single-mode analyses can provide information on the type of nonlinearity and can predict some of the non-linear phenomena exhibited in the response of shells to a harmonic excitation, such as, multiple solutions, jumps, and subharmonic and superharmonic resonances, they cannot predict combinational resonances and what is generally referred to as modal interactions[7]; the latter may provide a coupling or an energy exchange among the system's modes. This coupling can dominate the response of systems having

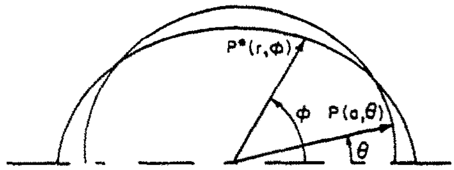

Fig. 1. Polar coordinates of a point on the shell which was initially at P and is at P* at time t^* (scale exaggerated).

some modes that are involved in internal (autoparametric) resonances, which occur when the linear natural frequencies are commensurable or nearly commensurable.

The first studies of modal interactions in the response of shells were initiated by McIvor[8, 9], Goodier and McIvor[10], McIvor and Sonstegard[11], and McIvor and Lovell[12]. They analyzed the response of cylindrical and spherical shells to radial and nearly radial impulses, taking into account the coupling of the breathing mode with a flexural mode when their frequencies are in the ratio of two-to-one. Integrating the governing ordinary-differential equations numerically they found that the energy is continuously exchanged between the internally resonant modes. They also linearized the equation governing the breathing mode and substituted the harmonic breathing response into the equation governing the flexural mode to obtain a Mathieu-type equation, the solutions of which indicate the regions of stability and instability of the breathing mode. Bieniek *et al.*[13] used Donnell's equations and determined an axially symmetric response. Then, they analyzed the stability of this response to asymmetric modes by deriving a Mathieu-type equation. Atluri[14] used the method of multiple scales to analyze free oscillations of shells in the absence of internal resonances. Mente[15] solved numerically a set of n non-linear equations arising from the Galerkin procedure. Chen and Babcock[16] investigated analytically and experimentally the non-linear response of cylindrical shells to a harmonic excitation. They studied both the driven as well as the companion mode and studied traveling waves.

Nayfeh and Raouf[17] analyzed the non-linear inextensional response of an infinitely long cylindrical shell to a harmonic excitation, when the frequency of the fundamental breathing mode is approximately twice the frequency of a flexural mode. They used the method of multiple scales to fully account for the non-linear interaction, including the influence of the flexural mode on the breathing mode. They demonstrated the existence of the saturation phenomenon in the response. In this paper, we relax the inextensional assumption and demonstrate the existence of a Hopf bifurcation, leading to amplitude- and phase-modulated rather than periodic motions. Moreover, we analyze the case in which the excitation frequency is near the frequency of the flexural mode and demonstrate the existence of a Hopf bifurcation in this case also.

2. PROBLEM FORMULATION

Following McIvor[8] and Goodier and McIvor[10], we consider the case of plane strain in which the strain parallel to the generators of the shell is everywhere zero. Thus, the deformation of the shell is identical in every plane perpendicular to the shell axis, and the shell can be considered as being in plane motion. In such a plane, we consider a point P on the undeformed shell midsurface with the polar coordinate (a, θ), which after a time t^* moves to P* with the polar coordinates r and ϕ, as shown in Fig. 1. We introduce the dimensionless displacement w and time t defined by

$$w = \frac{a-r}{a}, \quad t = \frac{ct^*}{a} \tag{1}$$

where t^* is the dimensional time, $c^2 = E/\rho(1-\nu^2)$, E is Young's modulus, ν is Poisson's ratio, and ρ is the density of the shell per unit width. Moreover, we let

$$\psi = \phi - \theta. \tag{2}$$

Then, to second order, the governing equations are[10, 17]

$$\ddot{w} + \alpha^2(w^{iv} + 2w'' + w) - \psi' + w = w''(\psi' - w) - \dot{\psi}^2 + \psi'^2 - 2w\psi' + w'\psi'' - \tfrac{1}{2}w'^2 + \frac{a(1-\nu^2)}{Eh}P(1 + \psi' - w) \tag{3}$$

and

$$\ddot{\psi} - \psi'' + w' = w'w'' - 2w'\psi' + 2\dot{w}\dot{\psi} + \frac{a(1-\nu^2)}{Eh}w'P \tag{4}$$

where the overdot indicates the partial derivative with respect to t, the prime indicates the partial derivative with respect to θ, and P is the applied pressure load. Here

$$\alpha^2 = h^2/12a^2 \tag{5}$$

where h and a are the thickness and initial radius of the shell.

Since the shell is closed, w and ψ must be periodic in θ and hence they can be expanded in a Fourier series as

$$w(\theta, t) = \eta_0(t) + \sum_{n=1}^{\infty} [\eta_n(t) \cos n\theta + \zeta_n(t) \sin n\theta] \tag{6}$$

$$\psi(\theta, t) = \xi_0(t) + \sum_{n=1}^{\infty} [-\xi_n(t) \cos n\theta + \chi_i(t) \sin n\theta]. \tag{7}$$

Goodier and McIvor[10] and Nayfeh and Raouf[17] used the inextensionality condition

$$w = \psi' \tag{8}$$

to relate ξ_n and χ_n in terms of η_n and ζ_n. The result is

$$\xi_n = \zeta_n/n \quad \text{and} \quad \chi_n = \eta_n/n. \tag{9}$$

Disregarding the rigid body rotation term ξ_0 and the rigid body translation terms η_1 and ζ_1, they substituted the assumed expressions for w and ψ into the kinetic and potential energies, carried out the θ integration, and used Lagrange's equations to obtain the non-linear equations that govern the η_0, η_n and ζ_n. Goodier and McIvor[10] restricted their analysis to impulses with durations much less than the period of the uniform radial mode of vibration. Such a restriction made it possible to convert the problem into that of free vibration. Moreover, they neglected the non-linear terms in the equation describing the breathing mode, which led to a harmonic expression for η_0, which upon substitution into the equation describing the flexural mode led to an equation with periodic coefficients. By analyzing this equation, they determined the conditions for the instability of the flexural mode. Moreover, they produced a numerical solution of the approximate equations of motion that accounts for the non-linear coupling of the breathing and flexural modes. They found that the energy is continuously exchanged between the two modes. Nayfeh and Raouf[17] used the method of multiple scales to determine a first-order uniform expansion for these equations in the case of a two-to-one internal resonance and a harmonic pressure loading having a frequency that is near the frequency of the breathing mode. In this paper, we relax the inextensionality assumption and also analyze the case where the frequency of the excitation is near the frequency of the flexural mode.

Substituting eqns (6) and (7) into eqns (3) and (4) and using the Galerkin procedure, one determines the non-linear equations that govern the η_n, ζ_n, ξ_n, and χ_n (see the Appendix). Approximate solutions of these equations can be obtained using a perturbation method, such as the method of multiple scales[18, 19]. In this paper, we apply the method of multiple scales directly to eqns (3) and (4).

3. PERTURBATION SOLUTION

We use the method of multiple scales[18, 19] to determine a second-order uniform expansion of the solution of eqns (3) and (4) for small but finite amplitudes when P is given by

$$\frac{a(1-\nu^2)}{Eh}P = \varepsilon\left[f_0 + \sum_{n=1}^{\infty} f_n \cos n\theta\right]\cos \Omega t \tag{10}$$

where ε is a small dimensionless quantity. Moreover, we seek a second-order uniform expansion in the form

$$w(\theta, t; \varepsilon) = \varepsilon w_1(\theta, T_0, T_1) + \varepsilon^2 w_2(\theta, T_0, T_1) + \cdots \tag{11}$$

$$\psi(\theta, t; \varepsilon) = \varepsilon\psi_1(\theta, T_0, T_1) + \varepsilon^2\psi_2(\theta, T_0, T_1) + \cdots \tag{12}$$

where $T_0 = t$, a fast scale characterizing motions with the natural and excitation frequencies, and $T_1 = \varepsilon t$, a slow scale characterizing the modulation of the amplitudes and phases of the modes with damping, nonlinearity, and any possible external resonances. In terms of T_0 and T_1, the time derivatives become

$$\frac{\partial}{\partial t} = D_0 + \varepsilon D_1 + \cdots \tag{13}$$

$$\frac{\partial^2}{\partial t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots \tag{14}$$

where $D_n = \partial/\partial T_n$. Substituting eqns (10)–(14) into eqns (3) and (4) and equating coefficients of like powers of ε, we obtain

order ε

$$D_0^2 w_1 + \alpha^2(w_1^{iv} + 2w_1'' + w_1) - \psi_1' + w_1 = 0 \tag{15}$$

$$D_0^2\psi_1 - \psi_1'' + w_1' = 0 \tag{16}$$

order ε^2

$$D_0^2 w_2 + \alpha^2(w_2^{iv} + 2w_2'' + w_2) - \psi_2 + w_2 = -2D_0D_1w_1 + w_1''(\psi_1' - w_1) - (D_0\psi_1)^2 + \psi_1'^2 - 2w_1\psi_1' + w_1'\psi_1'' - \tfrac{1}{2}w_1'^2 + \left[f_0 + \sum_{m=1}^{\infty} f_m \cos m\theta\right]\cos \Omega T_0 \tag{17}$$

$$D_0^2\psi_2 - \psi_2'' + w_2' = -2D_0D_1\psi_1 + w_1'w_1'' - 2w_1'\psi_1' + 2(D_0w_1)(D_0\psi_1). \tag{18}$$

Since the shell is closed, the w_n and ψ_n must be periodic in θ. Hence, the solution of eqns (15) and (16) is essentially the linear solution; that is

$$w_1 = A_0(T_1)\,e^{i\omega_0 T_0}+\sum_{m=1}^{\infty}[A_{m\omega}(T_1)\,e^{i\omega_m T_0}+A_{mp}(T_1)\,e^{ip_m T_0}]\cos m\theta + \sum_{m=1}^{\infty}[B_{m\omega}(T_1)\,e^{i\omega_m T_0}+B_{mp}(T_1)\,e^{ip_m T_0}]\sin m\theta+\text{cc} \quad (19)$$

$$\psi_1 = B_0(T_1)-\sum_{m=1}^{\infty}[\Gamma_{m\omega}B_{m\omega}\,e^{i\omega_m T_0}+\Gamma_{mp}B_{mp}\,e^{ip_m T_0}]\cos m\theta + \sum_{m=1}^{\infty}[\Gamma_{m\omega}A_{m\omega}\,e^{i\omega_m T_0}+\Gamma_{mp}A_{mp}\,e^{ip_m T_0}]\sin m\theta+\text{cc} \quad (20)$$

where cc stands for the complex conjugate of the preceding terms

$$\omega_0^2 = 1+\alpha^2 \quad (21)$$

and the frequencies ω_m and p_m are the roots of the frequency equation

$$\lambda^4-[m^2+1+\alpha^2(m^2-1)^2]\lambda^2+\alpha^2m^2(m^2-1)^2 = 0. \quad (22)$$

Moreover, the amplitude ratios are

$$\Gamma_{m\omega} = \frac{1}{m}[1+\alpha^2(m^2-1)^2-\omega_m^2] \quad (23)$$

$$\Gamma_{mp} = \frac{1}{m}[1+\alpha^2(m^2-1)^2-p_m^2]. \quad (24)$$

The A's and B's are arbitrary functions of T_1 to this order of approximation; they are determined by imposing the solvability conditions (elimination of secular and small-divisor terms) at the next level of approximation.

Substituting eqns (19) and (20) into eqns (17) and (18) yields two coupled inhomogeneous equations for the determination of w_2 and ψ_2. Any particular solution of these equations will contain secular or small-divisor terms if

(a) $\Omega \approx \lambda_m$: a condition of primary resonance
(b) $\lambda_m \approx 2\lambda_s$: a condition of two-to-one internal resonance
(c) $\lambda_m \approx \lambda_s+\lambda_r$: a condition of combination internal resonance

where the λ_i are the natural frequencies of the shell. In this paper, we treat the case in which $\omega_0 \approx 2\omega_n$, where ω_0 is the frequency of the breathing mode and ω_n is the frequency of the nth flexural mode. For this case of internal resonance, we analyze separately the cases of primary resonance of the breathing mode (i.e. $\Omega \approx \omega_0$) and primary resonance of the nth flexural mode (i.e. $\Omega \approx \omega_n$).

It turns out that, in the presence of structural or viscous damping, the A's and B's except A_0, B_0, $A_{n\omega}$, and $B_{n\omega}$ tend to zero as time tends to infinity. In other words, the free-oscillation terms of all modes that are not directly or indirectly excited vanish in the steady state. Consequently, we need only to include A_0, B_0, $A_{n\omega}$, and $B_{n\omega}$ in eqns (19) and (20). In what follows, we drop the subscript ω for convenience of notation. Thus, we replace eqns (19) and (20) with

$$w_1 = A_0(T_1)\,e^{i\omega_0 T_0}+A_n(T_1)\,e^{i\omega_n T_0}\cos n\theta+B_n(T_1)\,e^{i\omega_n T_0}\sin n\theta+\text{cc} \quad (25)$$

$$\psi_1 = B_0(T_1)-\Gamma_nB_n(T_1)\,e^{i\omega_n T_0}\cos n\theta+\Gamma_nA_n(T_1)\,e^{i\omega_n T_0}\sin n\theta+\text{cc}. \quad (26)$$

Substituting eqns (25) and (26) into eqns (17) and (18), we obtain

$$D_0^2 w_2+\alpha^2(w_2^{iv}+2w_2''+w_2)-\psi_2'+w_2 = -2i\omega_0 A_0' e^{i\omega_0 T_0}-2i\omega_n(A_n' \cos n\theta + B_n' \sin n\theta) e^{i\omega_n T_0}+[\tfrac{1}{4}n^2+\tfrac{1}{2}(\omega_n^2+n^2)\Gamma_n^2-n\Gamma_n](A_n^2+B_n^2) e^{2i\omega_n T_0} + (n^2-2n\Gamma_n)A_0(\bar{A}_n \cos n\theta+\bar{B}_n \sin n\theta) e^{i(\omega_0-\omega_n)T_0} \sin n\theta + \tfrac{1}{2}(f_0+f_n \cos n\theta) e^{i\Omega T_0}+\text{cc}+\text{NST} \tag{27}$$

$$D_0^2\psi_2-\psi_2''+w_2' = 2i\omega_n\Gamma_n(B_n' \cos n\theta-A_n' \sin n\theta) e^{i\omega_n T_0}-2\omega_0\omega_n\Gamma_n A_0(\bar{B}_n \cos n\theta - \bar{A}_n \sin n\theta) e^{i(\omega_0-\omega_n)T_0}+\text{cc}+\text{NST} \tag{28}$$

where NST stands for terms that do not produce secular or small-divisor terms.

Any particular solution of eqns (27) and (28) contains secular terms and small-divisor terms, depending on the resonant conditions. Therefore, to proceed further, we need to specify the resonances being considered. In the next section, we consider the case of primary resonance of the breathing mode (i.e. $\Omega \approx \omega_0$) in the presence of the internal resonant condition $\omega_0 \approx 2\omega_n$. In Section 4, we consider the case of primary resonance of the nth flexural mode in the presence of the same internal resonant condition.

4. PRIMARY RESONANCE OF THE BREATHING MODE

According to the method of multiple scales, we introduce the detuning parameter σ_1 and σ_2 to convert the small-divisor terms into secular terms as

$$\Omega = \omega_0+\varepsilon\sigma_1 \quad \text{and} \quad \omega_0 = 2\omega_n+\varepsilon\sigma_2 \tag{29}$$

and write

$$\Omega T_0 = \omega_0 T_0+\sigma_1 T_1 \quad \text{and} \quad \omega_0 T_0 = 2\omega_n T_0+\sigma_2 T_1. \tag{30}$$

We seek a particular solution of eqns (27) and (28) free of secular terms in the form

$$w_2 = R_0 e^{i\omega_0 T_0}+R_1 e^{i\omega_n T_0} \cos n\theta+R_2 e^{i\omega_n T_0} \sin n\theta+\text{cc} \tag{31}$$

$$\psi_2 = -Q_2 e^{i\omega_n T_0} \cos n\theta+Q_1 e^{i\omega_n T_0} \sin n\theta+\text{cc}. \tag{32}$$

Substituting eqns (31) and (32) into eqns (27) and (28), using eqns (30), and separating the θ and T_0 variations, we obtain

$$2iA_0'-4\Lambda_1(A_n^2+B_n^2) e^{-i\sigma_2 T_1}-f e^{i\sigma_1 T_1} = 0 \tag{33}$$

$$n\Gamma_n R_1-nQ_1 = -2i\omega_n A_n'+(n^2-2n\Gamma_n)A_0\bar{A}_n e^{i\sigma_2 T_1} \tag{34}$$

$$n\Gamma_n R_2-nQ_2 = -2i\omega_n B_n'+(n^2-2n\Gamma_n)A_0\bar{B}_n e^{i\sigma_2 T_1} \tag{35}$$

$$-nR_1+(n^2-\omega_n^2)Q_1 = -2i\omega_n\Gamma_n A_n'+2\omega_0\omega_n\Gamma_n A_0\bar{A}_n e^{i\sigma_2 T_1} \tag{36}$$

$$nR_2-(n^2-\omega_n^2)Q_2 = 2i\omega_n\Gamma_n B_n'-2\omega_0\omega_n\Gamma_n A_0\bar{B}_n e^{i\sigma_2 T_1} \tag{37}$$

where $f_0 = 2\omega_0 f$ and

$$4\omega_0\Lambda_1 = \tfrac{1}{4}n^2+\tfrac{1}{2}(\omega_n^2+n^2)\Gamma_n^2-n\Gamma_n. \tag{38}$$

Equation (33) provides one of the solvability conditions for eqns (27) and (28). The problem of determining the other two solvability conditions is transformed into the problem of determining the solvability conditions of eqns (34)–(37). Since the determinant of the coefficient matrix on the left-hand sides of eqns (34) and (36) vanishes on account of eqn

(22), these inhomogeneous equations have a solution if, and only if, a solvability condition is satisfied. This solvability condition can be obtained by adding eqn (34) to Γ_n times eqn (36). The result can be put in the form

$$2iA'_n - 4\Lambda_2 A_0 \bar{A}_n \, e^{i\sigma_2 T_1} = 0 \tag{39}$$

where

$$4\omega_n(1+\Gamma_n^2)\Lambda_2 = n^2 - 2n\Gamma_n + 2\omega_0\omega_n\Gamma_n^2. \tag{40}$$

Similarly, the solvability condition of eqns (35) and (37) can be put in the form

$$2iB'_n - 4\Lambda_2 A_0 \bar{B}_n \, e^{i\sigma_2 T_1} = 0. \tag{41}$$

Weak linear modal damping can be accounted for by modifying eqns (33), (39), and (41) as follows :

$$2i(A'_0 + \mu_0 A_0) - 4\Lambda_1(A_n^2 + B_n^2)\, e^{-i\sigma_2 T_1} - f\, e^{i\sigma_1 T_1} = 0 \tag{42}$$

$$2i(A'_n + \mu_n A_n) - 4\Lambda_2 A_0 \bar{A}_n \, e^{i\sigma_2 T_1} = 0 \tag{43}$$

$$2i(B'_n + \mu_n B_n) - 4\Lambda_2 A_0 \bar{B}_n \, e^{i\sigma_2 T_1} = 0 \tag{44}$$

where μ_0 is the damping coefficient for the breathing mode and μ_n is the damping coefficient for the nth flexural mode. To analyze the solutions of eqns (41)–(43), we express A_0, A_n, and B_n in the polar forms

$$A_0 = \tfrac{1}{2}a_0(T_1)\, e^{i\beta_0(T_1)} \tag{45}$$

$$A_n = \tfrac{1}{2}a_n(T_1)\, e^{i\beta_n(T_1)} \tag{46}$$

$$B_n = \tfrac{1}{2}b_n(T_1)\, e^{i\nu_n(T_1)} \tag{47}$$

where a_0 and β_0 are the amplitude and phase of the breathing mode, a_n and β_n are the amplitude and phase of the flexural mode, and b_n and ν_n are the amplitude and phase of the companion mode. Substituting eqns (45)–(47) into eqns (42)–(44) and separating real and imaginary parts, we obtain

$$a'_0 + \mu_0 a_0 + \Lambda_1 a_n^2 \sin\gamma_2 + \Lambda_1 b_n^2 \sin\gamma_3 - f\sin\gamma_1 = 0 \tag{48}$$

$$a'_n + \mu_n a_n - \Lambda_2 a_0 a_n \sin\gamma_2 = 0 \tag{49}$$

$$b'_n + \mu_n b_n - \Lambda_2 a_0 b_n \sin\gamma_3 = 0 \tag{50}$$

$$a_0\beta'_0 + \Lambda_1 a_n^2 \cos\gamma_2 + \Lambda_1 b_n^2 \cos\gamma_3 + f\cos\gamma_1 = 0 \tag{51}$$

$$a_n\beta'_n + \Lambda_2 a_0 a_n \cos\gamma_2 = 0 \tag{52}$$

$$b_n\nu'_n + \Lambda_2 a_0 b_n \cos\gamma_3 = 0 \tag{53}$$

where

$$\gamma_1 = \sigma_1 T_1 - \beta_0 \tag{54}$$

$$\gamma_2 = \beta_0 - 2\beta_n + \sigma_2 T_1 \tag{55}$$

$$\gamma_3 = \beta_0 - 2\nu_n + \sigma_2 T_1. \tag{56}$$

Fixed points and hence steady-state solutions of eqns (48)–(56) correspond to $a'_0 = a'_n = b'_n = 0$ and $\gamma'_n = 0$. It follows from eqns (54)–(56) that $\beta'_0 = \sigma_1$ and $\beta'_n = \nu'_n = \frac{1}{2}(\sigma_1 + \sigma_2)$. Hence, steady-state solutions correspond to the solutions of

$$\mu_0 a_0 + \Lambda_1 a_n^2 \sin \gamma_2 + \Lambda_1 b_n^2 \sin \gamma_3 - f \sin \gamma_1 = 0 \tag{57}$$

$$\mu_n a_n - \Lambda_2 a_0 a_n \sin \gamma_2 = 0 \tag{58}$$

$$\mu_n b_n - \Lambda_2 a_0 b_n \sin \gamma_3 = 0 \tag{59}$$

$$\sigma_1 a_0 + \Lambda_1 a_n^2 \cos \gamma_2 + \Lambda_1 b_n^2 \cos \gamma_3 + f \cos \gamma_1 = 0 \tag{60}$$

$$\tfrac{1}{2}(\sigma_1 + \sigma_2) a_n + \Lambda_2 a_0 a_n \cos \gamma_2 = 0 \tag{61}$$

$$\tfrac{1}{2}(\sigma_1 + \sigma_2) b_n + \Lambda_2 a_0 b_n \cos \gamma_3 = 0. \tag{62}$$

There are four possibilities. First

$$a_n = b_n = 0 \quad \text{and} \quad a_0 = f(\mu_0^2 + \sigma_1^2)^{-1/2} \tag{63}$$

which is essentially the linear solution. Second, $a_n = 0$ and $b_n \neq 0$. Third, $a_n \neq 0$ and $b_n = 0$. Fourth, $a_n \neq 0$ and $b_n \neq 0$. The last solution includes the second and third solutions as special cases. Then, it follows from eqns (58), (59), (61), and (62) that

$$a_0 = a_0^* = \Lambda_2^{-1}[\mu_n^2 + \tfrac{1}{4}(\sigma_1 + \sigma_2)^2]^{1/2} \tag{64}$$

$$\tan \gamma_2 = \tan \gamma_3 = -[2\mu_n/(\sigma_2 + \sigma_1)]. \tag{65}$$

Then, it follows from eqns (57) and (60) that

$$a_n^2 + b_n^2 = (\Lambda_1 \Lambda_2)^{-1}\{\tfrac{1}{2}\sigma_1(\sigma_1 + \sigma_2) - \mu_0 \mu_n \pm [f^2 \Lambda_2^2 - (\sigma_1 \mu_0 + \tfrac{1}{2}\mu_n(\sigma_1 + \sigma_2)^2)^2]^{1/2}\}. \tag{66}$$

Equation (64) shows that the amplitude a_0 of the directly excited breathing mode is independent of the amplitude f of the excitation. It depends only on the damping of the flexural mode and the detuning parameters σ_1 and σ_2. On the other hand, the amplitudes a_n and b_n of the flexural mode are strongly dependent on the excitation amplitude f.

To determine the stability of the steady-state solutions, we let

$$A_0 = \tfrac{1}{2}(p_1 - iq_1)\, e^{i\nu_1 T_1} \tag{67}$$

$$A_n = \tfrac{1}{2}(p_2 - iq_2)\, e^{i\nu_2 T_1} \tag{68}$$

$$B_n = \tfrac{1}{2}(p_3 - iq_3)\, e^{i\nu_2 T_1} \tag{69}$$

where

$$\nu_1 = \sigma_1 \quad \text{and} \quad \nu_2 = \tfrac{1}{2}(\sigma_1 + \sigma_2)$$

in eqns (42)–(44), separate real and imaginary parts, and obtain

$$p'_1 + \nu_1 q_1 + \mu_0 p_1 + 2\Lambda_1(p_2 q_2 + p_3 q_3) = 0 \tag{70}$$

$$q'_1 - \nu_1 p_1 + \mu_0 q_1 - \Lambda_1(p_2^2 + p_3^2 - q_2^2 - q_3^2) = f \tag{71}$$

$$p_2' + \nu_2 q_2 + \mu_n p_2 + \Lambda_2(q_1 p_2 - q_2 p_1) = 0 \tag{72}$$

$$q_2' - \nu_2 p_2 + \mu_n q_2 - \Lambda_2(p_1 p_2 + q_1 q_2) = 0 \tag{73}$$

$$p_3' + \nu_2 q_3 + \mu_n p_3 + \Lambda_2(q_1 p_3 - q_3 p_1) = 0 \tag{74}$$

$$q_3' - \nu_2 p_3 + \mu_n q_3 - \Lambda_2(p_1 p_3 + q_1 q_3) = 0. \tag{75}$$

Hence, the local stability of a fixed point with respect to a small perturbation proportional to $\exp(\lambda T_1)$ is determined by the zeros of the characteristic equation

$$\begin{vmatrix} \lambda+\mu_0 & \nu_1 & 2\Lambda_1 q_2 & 2\Lambda_1 p_2 & 2\Lambda_1 q_3 & 2\Lambda_1 p_3 \\ -\nu_1 & \lambda+\mu_0 & -2\Lambda_1 p_2 & 2\Lambda_1 q_2 & -2\Lambda_1 p_3 & 2\Lambda_1 q_3 \\ -\Lambda_2 q_2 & \Lambda_2 p_2 & \lambda+\mu_n+\Lambda_2 q_1 & \nu_2-\Lambda_2 p_1 & 0 & 0 \\ -\Lambda_2 p_2 & -\Lambda_2 q_2 & -\nu_2-\Lambda_2 p_1 & \lambda+\mu_n-\Lambda_2 q_1 & 0 & 0 \\ -\Lambda_2 q_3 & \Lambda_2 p_3 & 0 & 0 & \lambda+\mu_n+\Lambda_2 q_1 & \nu_2-\Lambda_2 p_1 \\ -\Lambda_2 p_3 & -\Lambda_2 q_3 & 0 & 0 & -\nu_2-\Lambda_2 p_1 & \lambda+\mu_n-\Lambda_2 q_1 \end{vmatrix} = 0. \tag{76}$$

To investigate the stability of the linear solution given by eqns (63), we put $p_2 = p_3 = q_2 = q_3 = 0$ in eqn (76) and, after some algebraic manipulations, obtain

$$[(\lambda+\mu_0)^2+\nu_1^2][(\lambda+\mu_n)^2+\nu_2^2-\Lambda_2^2 a_0^2]^2 = 0 \tag{77}$$

because $a_0^2 = p_1^2+q_1^2$. Hence

$$\lambda = -\mu_0 \pm i\nu_1,\quad -\mu_n \pm (\Lambda_2^2 a_0^2-\nu_2^2)^{1/2},\quad -\mu_n \pm (\Lambda_2^2 a_0^2-\nu_2^2)^{1/2}. \tag{78}$$

Consequently, the linear solution is stable if and only if

$$\Lambda_2^2 a_0^2 \leqslant \nu_2^2+\mu_n^2 \tag{79}$$

which, in conjunction with eqn (64), implies that the linear solution is stable if $a_0 \leqslant a_0^*$ and unstable if $a_0 > a_0^*$ or $f > f_2^* = a_0^*(\mu_0^2+\sigma_1^2)^{1/2}$.

To study the stability of the non-linear solution given by eqns (64)–(66) when $b_n = 0$, we let $p_3 = q_3 = 0$ in eqn (76), use eqn (64), and obtain

$$[(\lambda+\mu_n)^2-\mu_n^2]\{\lambda^4+2(\mu_0+\mu_n)\lambda^3+[\mu_0^2+4\mu_0\mu_n+\nu_1^2+4\Lambda_1\Lambda_2 a_n^2]\lambda^2 + [2\mu_n\mu_0^2+2\mu_n\nu_1^2+4\Lambda_1\Lambda_2(\mu_0+\mu_n)a_n^2]\lambda+4\Lambda_1\Lambda_2 a_n^2[\Lambda_1\Lambda_2 a_n^2+\mu_0\mu_n-\nu_1\nu_2]\} = 0. \tag{80}$$

Hence, either $\lambda = 0$ or $-2\mu_n$ or

$$\lambda^4+2(\mu_0+\mu_n)\lambda^3+[\mu_0^2+4\mu_0\mu_n+\nu_1^2+4\Lambda_1\Lambda_2 a_n^2]\lambda^2+[2\mu_n\mu_0^2+2\mu_n\nu_1^2+4\Lambda_1\Lambda_2(\mu_0+\mu_n)a_n^2]\lambda + 4\Lambda_1\Lambda_2 a_n^2[\Lambda_1\Lambda_2 a_n^2+\mu_0\mu_n-\nu_1\nu_2] = 0. \tag{81}$$

The necessary and sufficient conditions that none of the roots of eqn (81) has positive real parts then are

$$\Lambda_1\Lambda_2 a_n^2+\mu_0\mu_n-\nu_1\nu_2 > 0 \tag{82}$$

$$4\mu_0\mu_n(\mu_0^2+\nu_1^2)(4\mu_n^2+4\mu_0\mu_n+\mu_0^2+\nu_1^2)+8(\mu_0+\mu_n)^2\Lambda_1\Lambda_2 a_n^2(\mu_0^2+2\mu_0\mu_n+2\nu_1\nu_2+\nu_1^2) > 0. \tag{83}$$

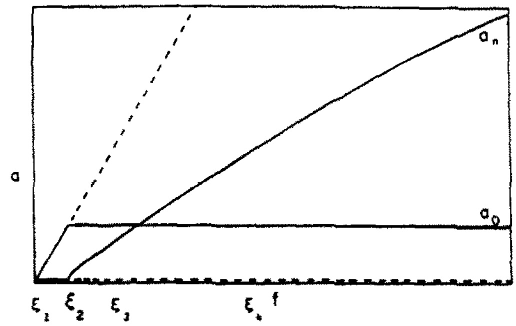

Fig. 2. Modal response amplitudes as functions of the amplitude of the excitation when $\Gamma < 0$.

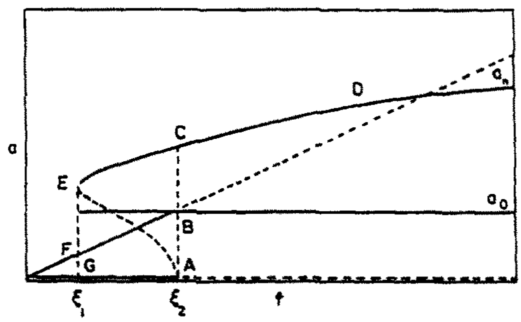

Fig. 3. Modal response amplitudes as functions of the amplitude of the excitation when $\Gamma > 0$.

Condition (82), in conjunction with eqn (66), implies that the solution corresponding to the positive sign is stable whereas the solution corresponding to the negative sign is unstable. The violation of condition (83) would imply the existence of a pair of complex-conjugate roots of eqn (81) with a positive real part, which in turn would imply the existence of a Hopf bifurcation. When $\nu_1\nu_2 > 0$, condition (83) is satisfied for all values of μ_0, μ_n, and f. On the other hand, when $\nu_1\nu_2 < 0$, condition (83) may be violated, depending on the values of μ_0, μ_n, and f.

Next, we present numerical results for the case $\alpha^2 = 2.0918 \times 10^{-4}$, which yields $\omega_0 = 1.0001$ and $\omega_6 = 0.4993$, $\Lambda_1 = 2.1257$ and $\Lambda_2 = 16.5653$. Thus, $\omega_0 \approx 2\omega_6$. We let $\mu_0 = 0.01$ and $\mu_6 = 0.01$.

In Fig. 2, we show a representative variation of the amplitudes of the breathing mode and nth flexural mode for the case $\Gamma < 0$, where

$$\Gamma = \tfrac{1}{2}\sigma_1(\sigma_1+\sigma_2)-\mu_0\mu_s.$$

If the shell is excited by a radial load of amplitude f and frequency $\Omega \approx \omega_0$, the linear solution shows that the steady-state amplitude a_6 of the flexural mode is zero, whereas the steady-state amplitude a_0 of the breathing mode increases linearly with f. However, including the non-linear terms shows that above a threshold value ξ_2 of f, where

$$\xi_2 = a_0^*(\mu_0^2+\sigma_1^2)^{1/2} = \Lambda_2^{-1}\{(\mu_0^2+\sigma_1^2)[\mu_6^2+\tfrac{1}{4}(\sigma_1+\sigma_2)^2]\}^{1/2}$$

the linear solution is unstable, a_0 remains constant (saturates), and the additional energy spills over into the flexural mode. If the excitation frequency is such that $\sigma_1 = \sigma_2$, then the threshold value ξ_2 of f becomes

$$\xi_2 = \mu_6\Lambda_2^{-1}(\mu_0^2+\sigma_1^2)^{1/2}$$

which can be very small, depending on the damping coefficients μ_0 and μ_6. Consequently, the linear solution is unstable and the shell responds nonlinearly even for small excitations.

In Fig. 3, we show a representative variation of the amplitudes of the breathing mode and the flexural mode (corresponding to $s = 6$) when $\Gamma > 0$. In addition to the saturation phenomenon, Fig. 3 exhibits the jump phenomenon. When the excitation amplitude f lies in the interval $[\xi_1, \xi_2]$, there are three possible steady-state responses. Two of these responses are stable: the trivial response and the larger amplitude response. The response that is attained physically depends on the initial conditions. If the excitation amplitude increases from zero, one observes only the breathing mode until f reaches ξ_2. As f increases beyond ξ_2, a_6 jumps from zero to point C, producing a large wrinkling of the shell. As f increases further, a_0 remains constant, whereas a_6 increases slowly along the curve ECD. If f decreases from a value corresponding to point D, a_6 decreases slowly along the curve DCE and a_0 remains constant until point E is reached. As f decreases below ξ_1, a_6 jumps down to zero and a_0 jumps down to point F. As f decreases further, a_6 remains zero and a_0 decreases linearly with f.

If the amplitude of the excitation is set at a value in the interval $[\xi_1, \xi_2]$ and the shell is initially undisturbed, the response corresponds to the linear solution, in which the shell is breathing without wrinkling. However, if the shell is disturbed, the shell may respond with the non-linear solution, in which the amplitude of the breathing mode as well as the flexural mode increase dramatically, yielding a much larger response.

The instability of the linear solution and the saturation phenomenon were first found analytically and verified numerically by Nayfeh *et al.*[20] in the response of ships. Later these phenomena were observed experimentally in the response of a simple model consisting of two beams and two concentrated masses by Haddow *et al.*[21] and in the non-linear vibration laboratory at VPI & SU.

5. PRIMARY RESONANCE OF THE FLEXURAL MODE

In this case, we introduce the detuning parameters σ_1 and σ_2 defined according to

$$\Omega = \omega_n + \varepsilon\sigma_1 \quad \text{and} \quad \omega_0 = 2\omega_n + \varepsilon\sigma_2 \tag{84}$$

and write

$$\Omega T_0 = \omega_n T_0 + \sigma_1 T_1 \quad \text{and} \quad \omega_0 T_0 = 2\omega_n T_0 + \sigma_2 T_1. \tag{85}$$

Carrying out an analysis similar to that in the preceding section and accounting for weak linear modal damping, we obtain the solvability conditions

$$2\mathrm{i}(A_0' + \mu_0 A_0) - 4\Lambda_1(A_n^2 + B_n^2)\, \mathrm{e}^{-\mathrm{i}\sigma_2 T_1} = 0 \tag{86}$$

$$2\mathrm{i}(A_n' + \mu_n A_n) - 4\Lambda_2 A_0 \bar{A}_n\, \mathrm{e}^{\mathrm{i}\sigma_2 T_1} - f\, \mathrm{e}^{\mathrm{i}\sigma_1 T_1} = 0 \tag{87}$$

$$2\mathrm{i}(B_n' + \mu_n B_n) - 4\Lambda_2 A_0 \bar{A}_n\, \mathrm{e}^{\mathrm{i}\sigma_2 T_1} = 0 \tag{88}$$

where $f_n = 2\omega_n f$ and Λ_1 and Λ_2 are defined in eqns (38) and (40). Substituting eqns (45)–(47) into eqns (86)–(88) and separating real and imaginary parts, we obtain

$$a_0' + \mu_0 a_0 + \Lambda_1 a_n^2 \sin\gamma_2 + \Lambda_1 b_n^2 \sin\gamma_3 = 0 \tag{89}$$

$$a_n' + \mu_n a_n - \Lambda_2 a_0 a_n \sin\gamma_2 - f \sin\gamma_1 = 0 \tag{90}$$

$$b_n' + \mu_n b_n - \Lambda_2 a_0 b_n \sin\gamma_3 = 0 \tag{91}$$

$$a_0 \beta_0' + \Lambda_1 a_n^2 \cos\gamma_2 + \Lambda_1 b_n^2 \cos\gamma_3 = 0 \tag{92}$$

$$a_n \beta_n' + \Lambda_2 a_0 a_n \cos\gamma_2 + f \cos\gamma_1 = 0 \tag{93}$$

$$b_n v'_n + \Lambda_2 a_0 b_n \cos \gamma_3 = 0 \tag{94}$$

where γ_2 and γ_3 are defined in eqns (55) and (56) and

$$\gamma_1 = \sigma_1 T_1 - \beta_n. \tag{95}$$

Fixed points and hence steady-state solutions of eqns (89)–(95) correspond to $a'_0 = a'_n = b'_n = 0$ and $\gamma'_n = 0$. It follows from eqns (55), (56) and (95) that $\beta'_n = \sigma_1$ and $\beta'_0 = 2\sigma_1 - \sigma_2$ and $v'_n = \sigma_1$. Hence, steady-state solutions correspond to the solutions of

$$\mu_0 a_0 + \Lambda_1 a_n^2 \sin \gamma_2 + \Lambda_1 b_n^2 \sin \gamma_3 = 0 \tag{96}$$

$$\mu_n a_n - \Lambda_2 a_0 a_n \sin \gamma_2 - f \sin \gamma_1 = 0 \tag{97}$$

$$\mu_n b_n - \Lambda_2 a_0 b_n \sin \gamma_3 = 0 \tag{98}$$

$$(2\sigma_1 - \sigma_2) a_0 + \Lambda_1 a_n^2 \cos \gamma_2 + \Lambda_1 b_n^2 \cos \gamma_3 = 0 \tag{99}$$

$$\sigma_1 a_n + \Lambda_2 a_0 a_n \cos \gamma_2 + f \cos \gamma_1 = 0 \tag{100}$$

$$\sigma_1 b_n + \Lambda_2 a_0 b_n \cos \gamma_3 = 0. \tag{101}$$

To determine the stability of the steady-state solutions, we substitute eqns (67)–(69), where $v_1 = \sigma_1 - 2\sigma_2$ and $v_2 = \sigma_1$, into eqns (86)–(88), separate real and imaginary parts, and obtain

$$p'_1 + v_1 q_1 + \mu_0 p_1 + 2\Lambda_1 (p_2 q_2 + p_3 q_3) = 0 \tag{102}$$

$$q'_1 - v_1 p_1 + \mu_0 q_1 - \Lambda_1 (p_2^2 + p_3^2 - q_2^2 - q_3^2) = 0 \tag{103}$$

$$p'_2 + v_2 q_2 + \mu_n p_2 + \Lambda_2 (q_1 p_2 - q_2 p_1) = 0 \tag{104}$$

$$q'_2 - v_2 p_2 + \mu_n q_2 - \Lambda_2 (p_1 p_2 + q_1 q_2) = f \tag{105}$$

$$p'_3 + v_2 q_3 + \mu_n p_3 + \Lambda_2 (q_1 p_3 - q_3 p_1) = 0 \tag{106}$$

$$q'_3 - v_2 p_3 + \mu_n q_3 - \Lambda_2 (p_1 p_3 + q_1 q_3) = 0. \tag{107}$$

Hence, the local stability of a fixed point with respect to a small perturbation proportional to exp (λT_1) is determined by the zeros of the characteristic equation, eqn (76).

An interesting case to be considered is the Hopf bifurcation phenomenon. In particular, we will consider the case where $\mu_0 = \mu_6 = 0.02$, $\sigma_1 = -0.1$ and $\sigma_2 = -0.18$, such input will assure the existence of a pair of complex conjugate roots to eqn (76) with positive real parts

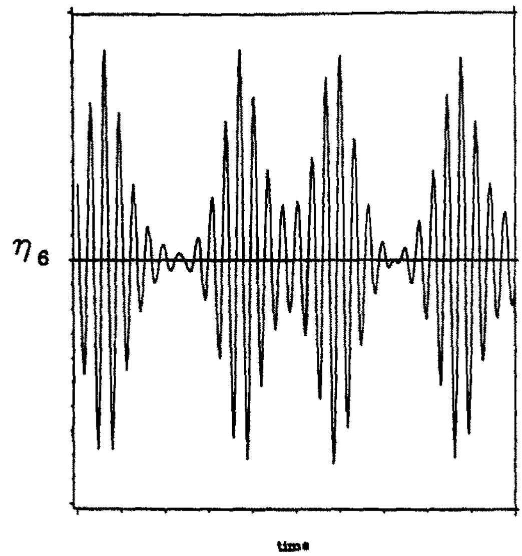

Fig. 4. Hopf bifurcation, a typical response history of the flexural mode.

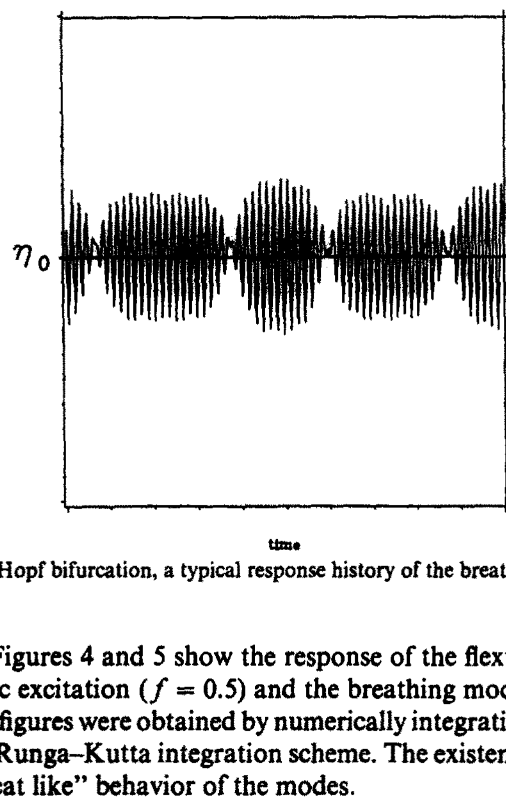

Fig. 5. Hopf bifurcation, a typical response history of the breathing mode.

(Hopf bifurcation). Figures 4 and 5 show the response of the flexural mode when directly excited by a harmonic excitation ($f = 0.5$) and the breathing mode, respectively, eqn (84) being satisfied. These figures were obtained by numerically integrating the original equations of motion through a Runga–Kutta integration scheme. The existence of a Hopf bifurcation is detected by the "beat like" behavior of the modes.

REFERENCES

1. E. H. Dowell and C. S. Ventres, Modal equations for the nonlinear flexural vibrations of a cylindrical shell. *Int. J. Solids Structures* **4**, 975–991 (1968).
2. J. H. Ginsberg, Nonlinear axisymmetric free vibration in simply supported cylindrical shells. *J. Appl. Mech.* **41**, 310–311 (1974).
3. L. H. Donnell, New theory of the buckling of thin cylindrical shells under axial compression and bending. *ASME J. Appl. Mech.* **56**, 795–806 (1934).
4. V. V. Novozhilov, *Foundations of the Nonlinear Theory of Elasticity*. Graylock Press, Rochester, New York (1953).
5. J. L. Sanders, Jr., Nonlinear theories of thin shells. *Q. Appl. Mech.* **XXI**, 21–36 (1963).
6. E. Reissner, On axi-symmetrical vibrations of shallow spherical shells. *Q. Appl. Math.* **13**, 279–290 (1955).
7. A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*. Wiley-Interscience, New York (1979).

8. I. K. McIvor, Dynamic stability and nonlinear oscillations of cylindrical shells (plane strain) subjected to impulsive pressure. Ph.D. Dissertation, Stanford University, Stanford, California (1962).
9. I. K. McIvor, The elastic cylindrical shell under radial impulse. *J. Appl. Mech.* **33**, 831–837 (1966).
10. J. N. Goodier and I. K. McIvor, The elastic cylindrical shell under nearly uniform radial impulse. *J. Appl. Mech.* **31**, 259–266 (1964).
11. I. K. McIvor and D. A. Sonstegard, Axisymmetric response of a closed spherical shell to a nearly uniform radial impulse. *J. Acoust. Soc. Am.* **40**, 1540–1547 (1966).
12. I. K. McIvor and E. G. Lovell, Dynamic response of finite-length cylindrical shells to nearly uniform radial impulse. *AIAA J.* **6**, 2346–2351 (1968).
13. M. P. Bieniek, T. C. Fan and L. M. Lackman, Dynamic stability of cylindrical shells. *AIAA J.* **4**, 495–500 (1966).
14. S. Atluri, A perturbation analysis of non-linear free flexural vibrations of a circular cylindrical shell. *Int. J. Solids Structures* **8**, 549–569 (1972).
15. L. J. Mente, Dynamic nonlinear response of cylindrical shells to asymmetric pressure loading. *AIAA J.* **11**, 793–800 (1973).
16. J. C. Chen and C. D. Babcock, Nonlinear vibration of cylindrical shells. *AIAA J.* **13**, 868–876 (1975).
17. A. H. Nayfeh and R. A. Raouf, Nonlinear forced response of circular cylindrical shells, presented at the ASME 1986 Pressure Vessel and Piping Conference, Chicago, Illinois, 20–24 July (1986).
18. A. H. Nayfeh, *Perturbation Methods*. Wiley-Interscience, New York (1973).
19. A. H. Nayfeh, *Introduction to Perturbation Techniques*. Wiley-Interscience, New York (1981).
20. A. H. Nayfeh, D. T. Mook and L. R. Marshall, Nonlinear coupling of pitch and roll modes in ship motion. *J. Hydronautics* **7**, 145–152 (1973).
21. A. G. Haddow, A. D. S. Barr and D. T. Mook, Theoretical and experimental study of modal interaction in a two-degree-of-freedom structure. *J. Sound Vibr.* **97**, 451–473 (1984).

APPENDIX

$$\ddot{\eta}_0+(1+\alpha^2)\eta_0 = -\dot{\xi}^2+\sum_m[\tfrac{1}{4}n^2(\eta_n^2+\zeta_n^2)+\tfrac{1}{2}n^2(\xi_n^2+\chi_n^2)-\tfrac{1}{2}(\dot{\xi}_n^2+\dot{\chi}_n^2)-n(\xi_n\zeta_n+\chi_n\eta_n)]$$
$$+f_0-f_0\eta_0+\sum_m[f_m(m\chi_m-\eta_m)+g_m(m\xi_m-\zeta_m)] \quad \text{(A1)}$$

$$\ddot{\eta}_n+[1+\alpha^2(n^2-1)^2]\eta_n-n\chi_n = n^2\eta_0\eta_n-2n\chi_n\eta_0+2\dot{\xi}_0\dot{\xi}_n+f_n+f_0(n\chi_n-\eta_n)-f_n\eta_0 \quad \text{(A2)}$$

$$\ddot{\zeta}_n+[1+\alpha^2(n^2-1)^2]\zeta_n-n\xi_n = n^2\eta_0\zeta_n-2n\eta_0\xi_n-2\dot{\xi}_0\dot{\chi}_n+g_n+f_0(n\xi_n-\zeta_n)-g_n\eta_0 \quad \text{(A3)}$$

$$\ddot{\xi}_n+n^2\xi_n-n\zeta_n = 2\dot{\eta}_0\dot{\xi}_n-2\dot{\xi}_0\dot{\eta}_n nf_0\zeta_n \quad \text{(A4)}$$

$$\ddot{\chi}_n+n^2\chi_n-n\eta_n = 2\dot{\eta}_0\dot{\chi}_n+2\dot{\xi}_0\zeta_n-nf_0\eta_n \quad \text{(A5)}$$

$$\ddot{\xi}_0 = 2\dot{\xi}_0\dot{\eta}_0-n^2(\chi_n\zeta_n-\xi_n\eta_n)-\dot{\xi}_n\dot{\eta}_n+\dot{\chi}_n\dot{\zeta}_n \quad \text{(A6)}$$

where

$$\frac{a(1-\nu^2)P}{Eh} = f_0-\sum(f_n\cos n\theta+g_n\sin n\theta). \quad \text{(A7)}$$